

# 12

## Tessellations

The Dutch artist M.C. Escher was fascinated by tiling patterns, also called tessellations. Escher made these tiling patterns by starting with a basic shape and then transforming the shape using translations, rotations, and reflections. These tessellations were very complex and many of them looked like animals and humans.

Escher created this tessellation by translating a parallelogram with griffins drawn on it. A griffin (or gryphon) is a legendary creature with the body of a lion and the head and wings of an eagle. Since the lion was considered the “king of beasts” and the eagle the “king of the air,” the griffin was thought to be an especially powerful and majestic creature.

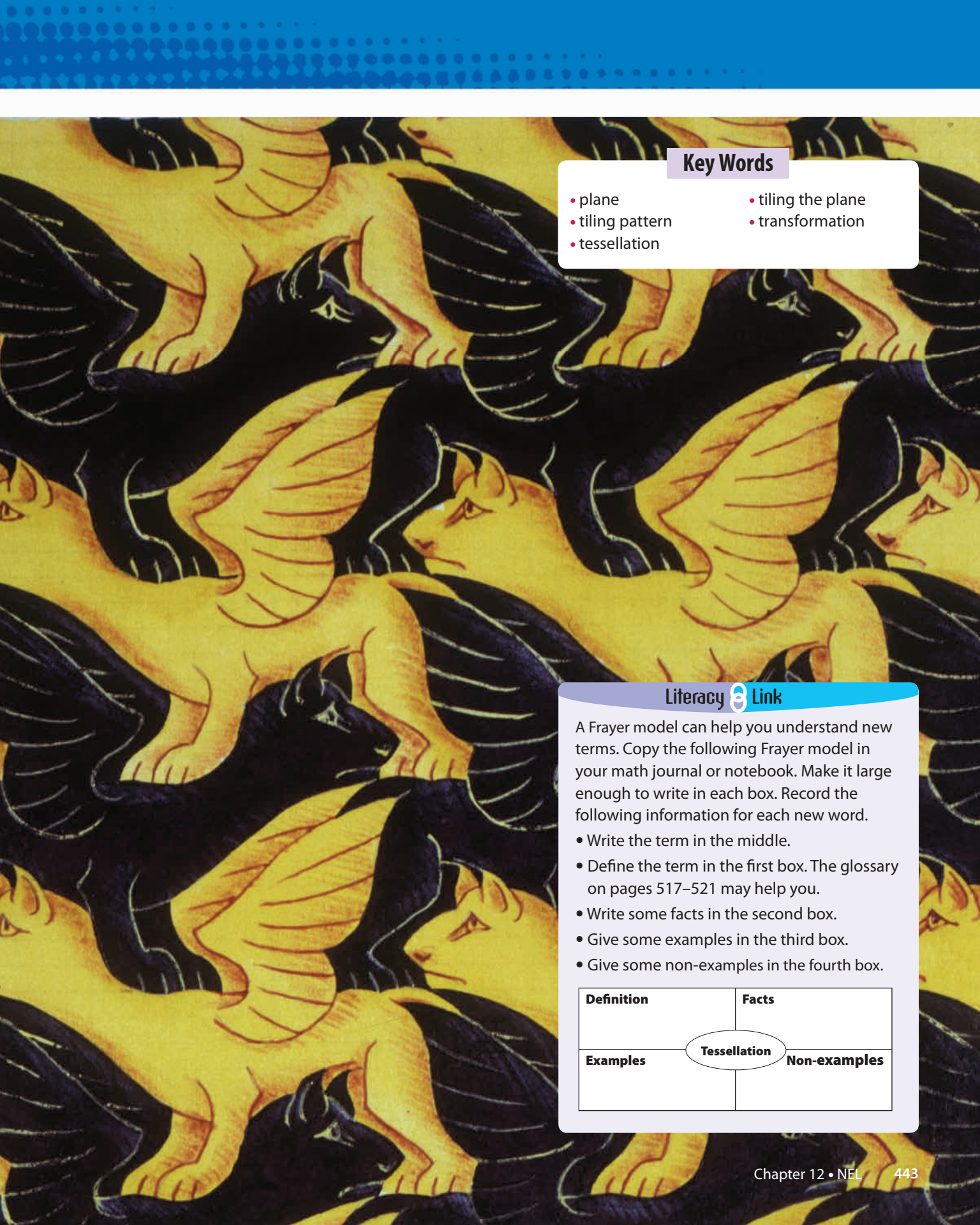
In this chapter, you will learn how to describe and create tessellations.

### What You Will Learn

- to describe and create tessellations
- to explore and describe tessellations in the environment







### Key Words

- plane
- tiling pattern
- tessellation
- tiling the plane
- transformation

### Literacy Link

A Frayer model can help you understand new terms. Copy the following Frayer model in your math journal or notebook. Make it large enough to write in each box. Record the following information for each new word.


- Write the term in the middle.
- Define the term in the first box. The glossary on pages 517–521 may help you.
- Write some facts in the second box.
- Give some examples in the third box.
- Give some non-examples in the fourth box.

|                   |                     |
|-------------------|---------------------|
| <b>Definition</b> | <b>Facts</b>        |
| <b>Examples</b>   | <b>Non-examples</b> |

**Tessellation**

## Making the Foldable

### Materials

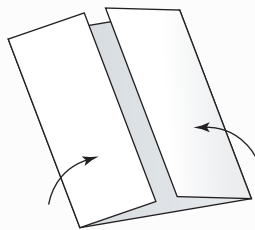
- 11 by 17 sheet of paper
- three sheets of 0.5-cm grid paper 
- scissors
- stapler
- glue (optional)

### Step 1

Fold an 11 × 17 sheet of paper in half lengthwise and pinch the centre to show the midpoint.

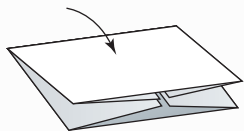
### Step 2

Fold the outside edges toward the centre.



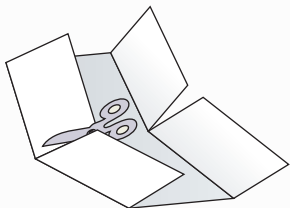
### Step 3

Fold the paper in half again the other way.



### Step 4

Cut along the crease to create four doors.

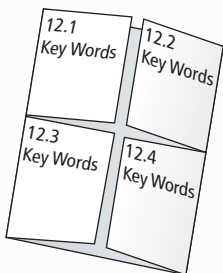


### Step 5

Label each door as shown.

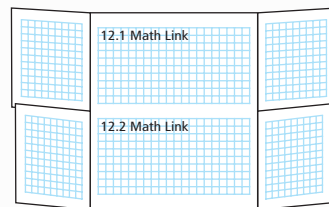
### Step 6

Cut pieces of 0.5-cm grid paper the size of each door flap and glue or staple them on the inside of each flap.



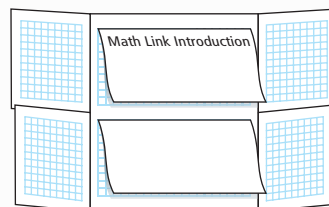
### Step 7

Cut a full sheet of 0.5-cm grid paper in half. Label the pieces as shown. Place the pieces in the middle section behind the doors you labelled in Step 5.



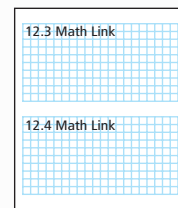
### Step 8

Cut a full sheet of blank notebook paper in half. Staple each piece on top of the Math Link grids for 12.1 and 12.2, as shown.



### Step 9

Cut another sheet of 0.5-cm grid paper in half horizontally. Staple these to the back of the Foldable and label them as shown.



## Using the Foldable

As you work through each section of Chapter 12, list and define the Key Words on the outside of the flap for each section. Place and label examples on the inside of the flap for each section.

Record your answers to the Math Link introduction on page 445 on the blank sheets inside the Foldable. Use the grids inside the Foldable and on the back to keep track of the designs you develop for each Math Link during the chapter.

In the space underneath each Math Link grid, make notes under the heading What I Need to Work On. Check off each item as you deal with it.

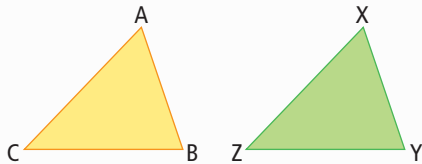


## MATH LINK

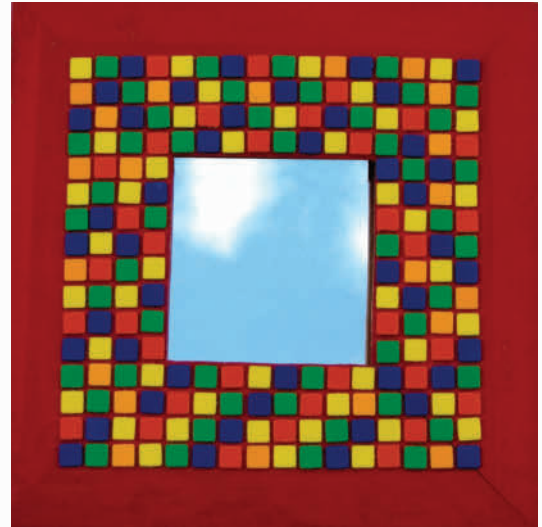
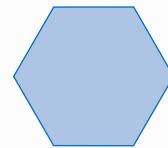
### Mosaic Designs

Mosaics can be used to decorate shelves, table tops, mirrors, floors, walls, and other objects. In this chapter, you will learn how to design and make your own mosaic.

1. Irregularly shaped triangular pieces can be used to create mosaics. What makes a triangle *irregular* in shape?
2. **a)** If triangular tiles are congruent, they can be used to make a mosaic. How can you tell if the triangular tiles labelled ABC and XYZ are congruent or not? Explain your reasoning.



- b)** Copy the shape of one of the triangles onto a piece of cardboard or construction paper. Cut out the triangle. Create a design on half of a blank sheet of paper by repeatedly tracing the triangle. Make sure that the sheet of paper is covered and there are no spaces left between the triangles.
  - c)** Colour your design.
3. **a)** Regular polygons can also be used to create interesting mosaics. What characteristics make a polygon *regular*?  
**b)** Copy the shape of this regular hexagon onto a piece of cardboard or construction paper. Cut out the hexagon. Create a new design using the same process you used for the irregular triangles in #2b).  
**c)** Write a brief paragraph explaining what geometric transformations you used to create your design in part b) of this question. For example, did you use translations, rotations, or reflections to make your design? Did you use a combination of transformations? If so, what steps did you follow to create your design?



#### Did You Know?

A tile used to make a mosaic is called a *tessera*. This word comes from the ancient Greek word *tessares*, which means *four*. The tiles used to make ancient mosaics had four corners.

# 12.1

## Exploring Tessellations With Regular and Irregular Polygons

### Focus on...

After this lesson, you will be able to...

- identify regular and irregular polygons that can be used to create tessellations
- describe why certain regular and irregular polygons can be used to tessellate the plane
- create simple tessellating patterns using polygons

Mosaics are often made of repeating patterns of tiles. What patterns do you see in the design?

Many mosaic tile designs are made from shapes that cover the area, or the plane, without overlapping or leaving gaps. These patterns are called **tiling patterns** or **tessellations**. Covering the plane in this way is called **tiling the plane**.



### Explore the Math

Which shapes can you use to tile or tessellate the plane?

1. Copy the following table into your notebook.

| Shape                   | Regular or Irregular Polygon? | Measure of Each Interior Angle | Prediction: Will the shape tile the plane? | Result: Does the shape tile the plane? |
|-------------------------|-------------------------------|--------------------------------|--|--|
| Equilateral triangle    |                               |                                |  |  |
| Isosceles triangle      |                               |                                |  |  |
| Square                  |                               |                                |  |  |
| Regular pentagon        |                               |                                |  |  |
| Regular hexagon         |                               |                                |  |  |
| Regular octagon         |                               |                                |  |  |
| Irregular quadrilateral |                               |                                |  |  |
| Irregular pentagon      |                               |                                |  |  |
| Irregular hexagon       |                               |                                |  |  |

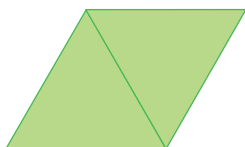
#### tiling pattern

- a pattern that covers an area or plane without overlapping or leaving gaps
- also called a **tessellation**

#### tiling the plane

- using repeated congruent shapes to cover an area without leaving gaps or overlapping
- also called tessellating the plane

2. a) Select an equilateral triangle block. Is this a regular or irregular polygon? Record your answer in the table.
  - b) Measure each interior angle and record your measurements in the table.
  - c) Predict whether the shape will tile the plane. Record your prediction in the table.
3. Trace the outline of the equilateral triangle. Move the triangle to a new position, so that the two triangles share a common side. Trace the outline of the triangle again. Continue to see if the shape tiles the plane. Record your conclusion in the table.



4. Use the same method to find out if the isosceles triangle, square, regular pentagon, regular hexagon, and regular octagon tile the plane. Record your results in the table.
5. Cut out the shape of an irregular quadrilateral.
  - a) Predict whether the shape will tile the plane.
  - b) Try to tile the plane with the shape. Record your results in the table.
  - c) Repeat steps 5a) and 5b) using an irregular pentagon and an irregular hexagon of your own design.

### Materials

- set of pattern blocks, or cardboard cutouts of pattern block shapes
- protractor
- cardboard cutouts of an isosceles triangle, a regular pentagon, and a regular octagon
- cardboard
- scissors
- ruler

*penta means 5  
hexa means 6  
octa means 8*

### Reflect on Your Findings

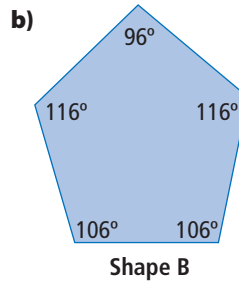
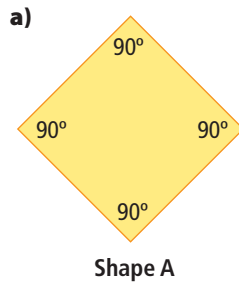
6. a) What regular shapes tile the plane? Explain why some regular shapes tile the plane but others do not. Hint: Look at the interior angle measures. Is there a pattern?
  - b) Explain why some irregular shapes tile the plane but others do not.

### Literacy Link

The term *plane* means a two-dimensional flat surface that extends in all directions.

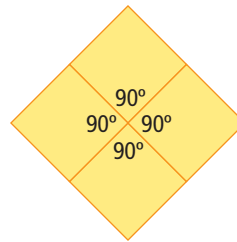
## Example: Identify Shapes That Tessellate the Plane

Do these polygons tessellate the plane? Explain why or why not.



### Solution

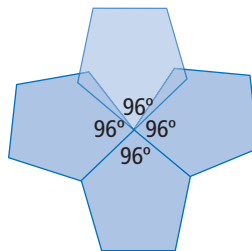
- a)** Arrange the squares along a common side. The rotated squares do not overlap or leave gaps when you try to form them into a tessellation. Shape A can be used to tessellate the plane.



Check:

Each of the interior angles where the vertices of the polygons meet is  $90^\circ$ . The sum of the four angles is  $90^\circ + 90^\circ + 90^\circ + 90^\circ = 360^\circ$ . This is equal to a full turn. The shape can be used to tessellate the plane.

- b)** Arrange the pentagons along a common side. The irregular pentagons overlap or leave gaps when you try to form them into a tessellation. Shape B cannot be used to tessellate the plane.



What other possible arrangements of the pentagons can you find? Do they overlap or leave gaps?

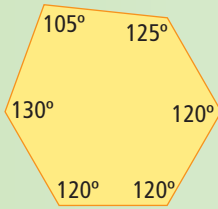
Check:

Each of the interior angles where the vertices of the polygons meet is  $96^\circ$ . The sum of the four angles is  $96^\circ + 96^\circ + 96^\circ + 96^\circ = 384^\circ$ . This is more than a full turn. The shape cannot be used to tessellate the plane.

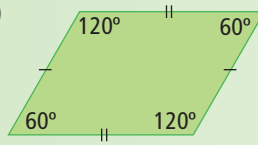
## Show You Know

Which of the following shapes can be used to tessellate the plane? Explain your reasoning.

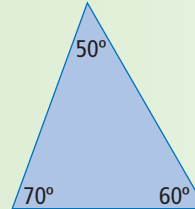
a)



b)

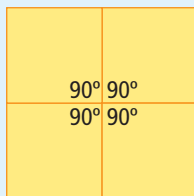


c)

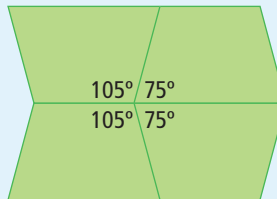


## Key Ideas

- A tiling pattern or tessellation is a pattern that covers a plane without overlapping or leaving gaps.
- Only three types of regular polygons tessellate the plane.
- Some types of irregular polygons tessellate the plane.
- Regular and irregular polygons tessellate the plane when the interior angle measures total exactly  $360^\circ$  at the point where the vertices of the polygons meet.



$$90^\circ + 90^\circ + 90^\circ + 90^\circ = 360^\circ$$



$$105^\circ + 75^\circ + 75^\circ + 105^\circ = 360^\circ$$

## Communicate the Ideas

1. Draw three types of regular polygons that tessellate the plane. Justify your choices.
2. What are two types of irregular polygons that can be used to tessellate the plane? Explain your choices to a friend.
3. Megan is tiling her kitchen floor. Should she choose ceramic tiles in the shape of a regular octagon? Explain how you know.



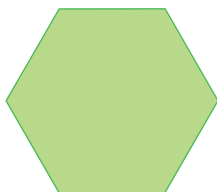
# Check Your Understanding

## Practise

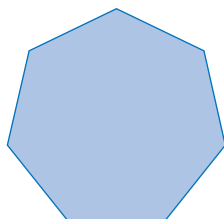
For help with #4 to #7, refer to the Example on page 448.

4. Do these regular polygons tessellate the plane? Explain why or why not.

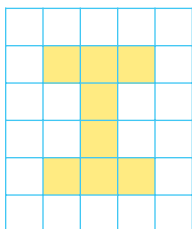
a)



b)



5. Use this shape to tessellate the plane. Show and colour the result on grid paper.

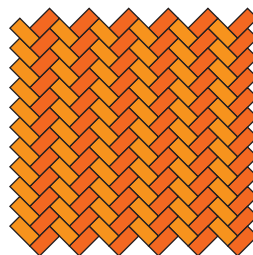


6. Tessellate the plane with an isosceles triangle. Use colours or shading to create an interesting design on grid paper.
7. Describe three tessellating patterns that you see at home or at school. What shapes make up the tessellation?

## Apply

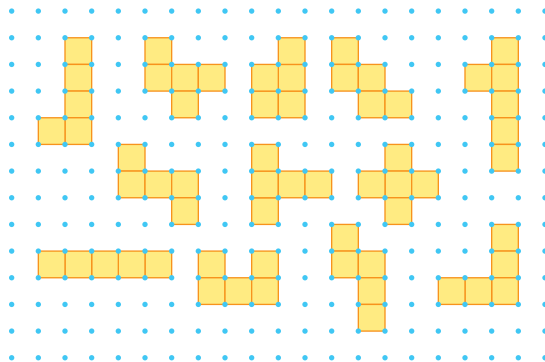
8. Jared is painting a mosaic on one wall of her bedroom that is made up of tessellating equilateral triangles. Describe two different tessellation patterns that Jared could use. Use triangular dot paper to help you describe the tessellations.

9. Patios are often made from interlocking rectangular bricks. The pattern shown below is called herringbone.



On grid paper, create two different patio designs from congruent rectangular bricks.

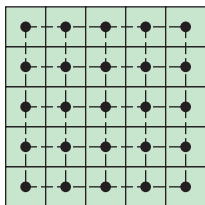
10. Some pentagons can be used to tessellate the plane.
- a) Describe a pentagon that will tessellate the plane. Explain how it tessellates the plane.
- b) Compare your pentagon with those of your classmates. How many different tessellating pentagons did you and your classmates find?
11. A pentomino is a shape made up of five squares. Choose two of the following pentominoes and try to make a tessellation with each one. Do each of your pentominoes make a tessellation? Explain why or why not.



12. Sarah is designing a pattern for the hood and cuffs of her new parka. She wants to use a regular polygon in the design and three different colours. Use grid paper to create two different designs that Sarah might use. Colour your designs.

### Extend

13. The diagram shows a tessellation of squares. A dot has been added to the centre of each square. The dots are joined by dashed segments perpendicular to common sides. The result is another tessellation, which is called the *dual* of the original tessellation.



- a) Describe the dual of the original square tessellation.  
 b) Draw a tessellation of regular hexagons. Draw and describe its dual.  
 c) Draw a tessellation of equilateral triangles. Draw and describe its dual.

14. Identify two different regular polygons that can be used together to create a tessellating pattern. Draw a tessellation on grid paper using the two polygons.

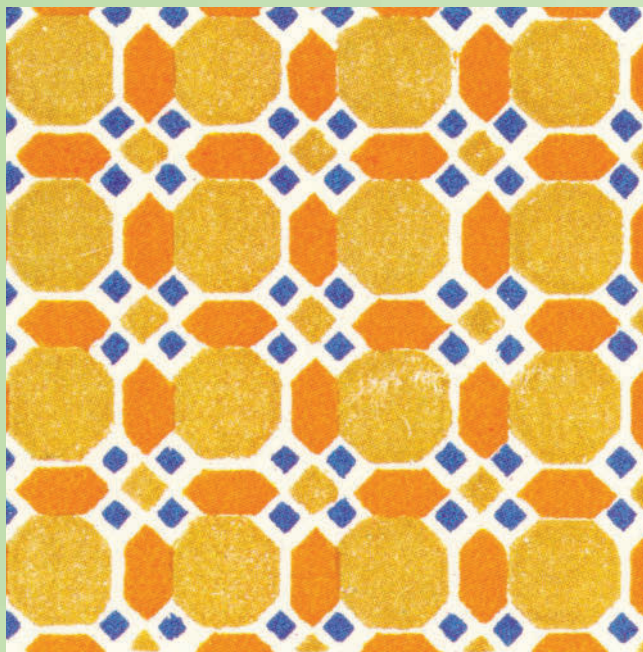
### Did You Know?

Many Islamic artists make very intricate geometric decorations and are experts at tessellation art.

## MATH LINK

This tiling pattern is from Alhambra, a Moorish palace built in Granada, Spain. Four different tile shapes are used to create this pattern.

- a) Describe the four shapes. Are they regular or irregular polygons?  
 b) Use templates to trace the shapes onto cardboard or construction paper.  
 c) Cut out ten of each shape and use some or all of them to create at least two different tile mosaics. Use each of the four shapes in your mosaics.



### WWW Web Link

To generate tessellations on the computer, go to [www.mathlinks8.ca](http://www.mathlinks8.ca) and follow the links.

# 12.2

## Constructing Tessellations Using Translations and Reflections

### Focus on...

After this lesson, you will be able to...

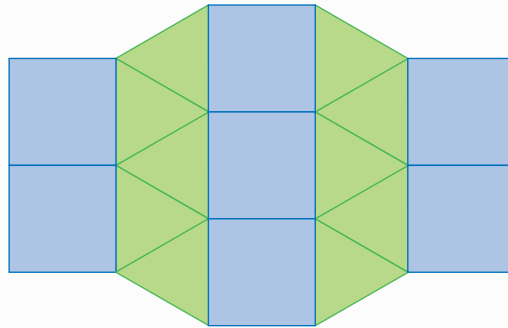
- identify how translations and reflections can be used to create a tessellation
- create tessellating patterns using two or more polygons

### transformation

- a change in a figure that results in a different position or orientation

### Materials

- set of pattern blocks, or cardboard cutouts of pattern block shapes
- ruler
- scissors
- glue stick
- tape
- cardboard or construction paper

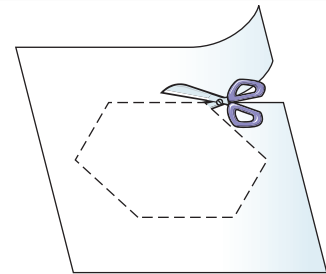


In section 12.1 you created simple tessellating patterns using regular and irregular polygons. Tessellations can also be made by combining regular or irregular polygons and then transforming them. Do you recognize the polygons used in this tessellation? What **transformations** were used to create the pattern?

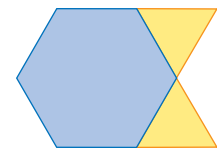
### Explore the Math

#### How can you create a tessellation using transformations?

1. Draw a regular hexagon on a piece of paper using a pattern block or cardboard cutout. Cut out the hexagon and glue it to a sheet of cardboard or construction paper.



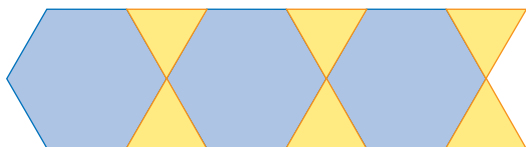
2. Draw two equilateral triangles on a piece of paper using a pattern block or cardboard cutout. Make sure that the side lengths of the triangles are the same as the side lengths of the hexagon. Cut out the triangles and glue them to a sheet of cardboard or construction paper so that they are attached to the sides of the hexagon as shown.



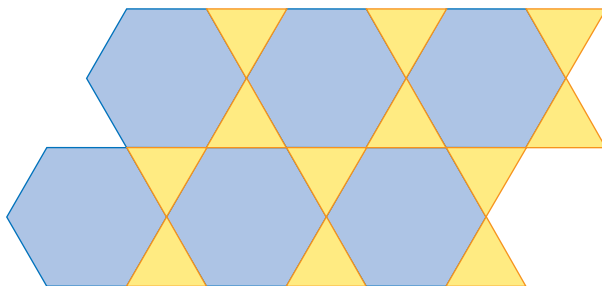
3. Cut out the combined shape. Trace the shape on a new sheet of paper.



4. Translate the shape so that the hexagon fits into the space formed by the two triangles. Trace around the translated shape and repeat two more times. What other ways can you translate the shape?



5. Translate the combined piece vertically and horizontally so that the base of the hexagon is now at the top of one of the triangles.

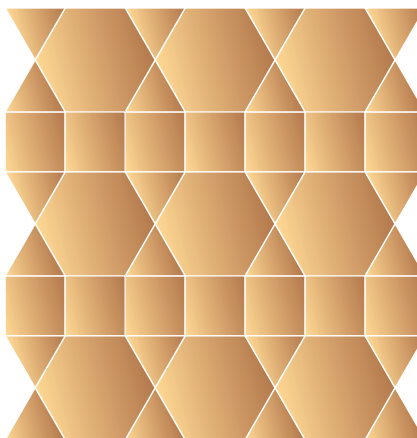


### Reflect on Your Findings

6. a) Describe how to use translations to create tessellations.  
 b) What other transformations could you use to get the same pattern as in #5? Explain the difference.

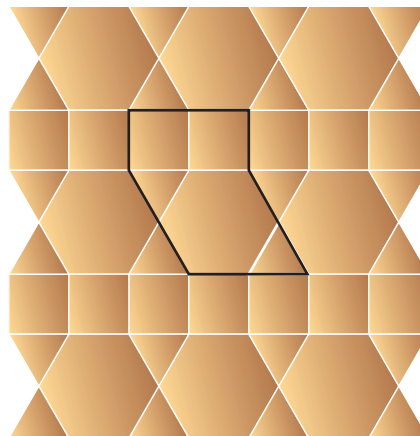
### Example: Identify the Transformation

- a) What polygons and what transformations are used to create this tessellation?  
 b) Does the area of the tessellating tile change during the tessellation?



### Solution

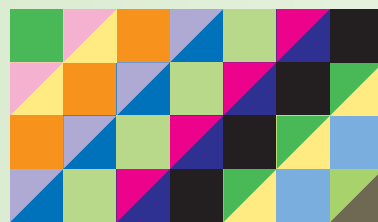
- a) The tessellation is made from a tessellating tile consisting of a hexagon with two squares and two equilateral triangles. The tessellating tile is then translated vertically and horizontally. This tessellation is created using translations.



- b) The area of the tessellating tile remains the same throughout the tessellation. There are no gaps or overlapping pieces.

### Show You Know

What transformation was used to create this tessellation? Explain your reasoning.



## Key Ideas

- Tessellations can be made with two or more polygons as long as the interior angles where the vertices of the polygons meet total exactly  $360^\circ$ .
- Two types of transformations commonly used to create tessellations are
  - translations
  - reflections
- The area of the tessellating tile remains the same after it has been transformed to create a tessellation.

### Communicate the Ideas

1. Brian missed today's class. How would you explain to him why some tessellating patterns made using translations could also be made using reflections?

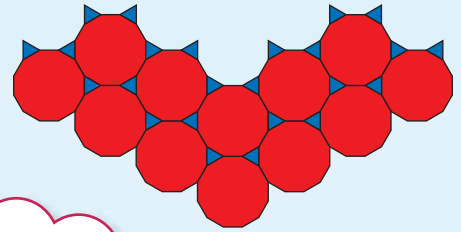
2. Ashley and Vijay are trying to figure out how this tessellation was made. Whose answer is correct? Explain.

Ashley says:

The tessellation is based on reflecting the blue triangles across the red dodecagon.

Vijay says:

The tessellation is based on translating the red dodecagon with 2 blue triangles.



**Literacy Link**

A dodecagon is a 12-sided polygon.

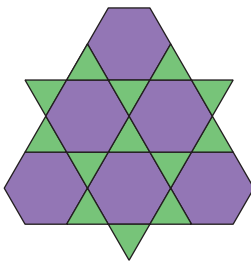
## Check Your Understanding

### Practise

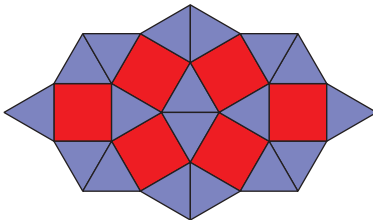
For help with #3 and #4, refer to the Example on pages 453–454.

3. Identify the two regular polygons used to create each tessellation.

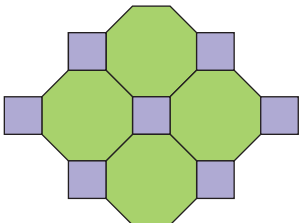
a)



b)



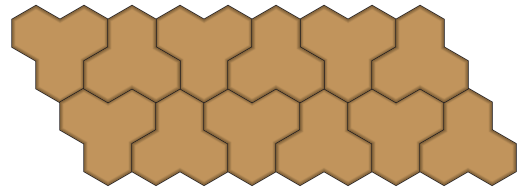
c)



4. What type of transformation could be used to create each tessellation in #3?

### Apply

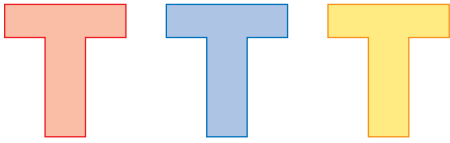
5. The diagram shows a garden path made from irregular 12-sided bricks.



- Explain why the 12-sided brick tessellates the plane.
- Use grid paper to design an irregular ten-sided brick that could be used to make a path.
- Explain why your ten-sided brick tessellates the plane.
- Use grid paper to design an irregular six-sided brick that could be used to tessellate the plane.
- Explain why your six-sided brick tessellates the plane.

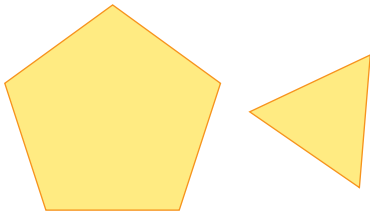


6. Simon is designing a wallpaper pattern that tessellates. He chooses to use the letter “T” as the basis of his pattern. Create two tessellations using the three coloured letters shown.



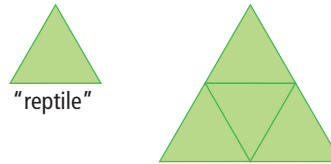
7. Priya is designing a kitchen tile that uses two different regular polygons. She then uses two different translations to create a tessellation. Use grid paper to design a tile that Priya could use. Show how it tiles the plane.

8. Barbara wants to make a quilt using the two polygons shown. Will she be able to create a tessellating pattern using these shapes? Explain.

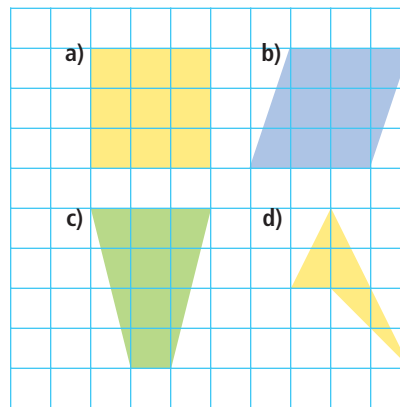


### Extend

9. An equilateral triangle is called a reptile (an abbreviation for “repeating tile”) because four equilateral triangles can be arranged to form a larger equilateral triangle.



Which of these figures are reptiles? Use grid paper to draw the larger figure for each reptile.



## MATH LINK

Many quilt designs are made using tessellating shapes. This quilt uses fabric cut into triangles that are sewn together to form squares. The squares are then translated vertically and horizontally.

Design your own quilt square using one or more regular tessellating polygons. Create an interesting design based on patterns or colours.



# 12.3

## Constructing Tessellations Using Rotations

### Focus on...

After this lesson, you will be able to...

- identify how rotations can be used to create a tessellation
- create tessellating patterns using two or more polygons

### Did You Know?

Professor Ronald Resch of the University of Utah built the world's largest pysanka from 3500 pieces of aluminum. It is located in Vegreville, Alberta; weighs 2300 kg; is 9.4 m high, 7 m long, and 5.5 m wide; and turns in the wind like a weather vane!



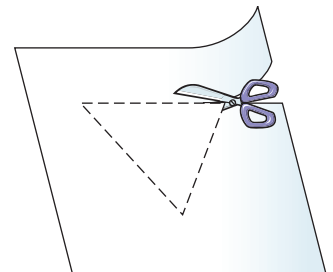
Pysanky is the ancient Eastern European art of egg decorating. The Ukrainian version of pysanka is the most well known. The name comes from the verb *to write*, because artists use a stylus to write with wax on the eggshell. Can you see how rotations are used to make the patterns on these eggs?



### Explore the Math

#### How can you create tessellations using rotations?

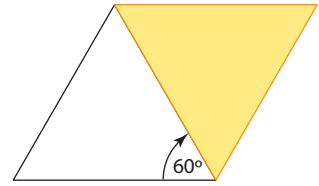
1. Draw an equilateral triangle with side lengths of 4 to 5 cm on a piece of paper. Cut out the triangle and glue it to a sheet of cardboard or construction paper to create a tile.
2. Trace around your tile on a piece of paper.



### Materials

- tracing paper
- scissors
- glue stick
- tape
- cardboard or construction paper
- coloured pencils

3. Rotate the tile  $60^\circ$  about one vertex until the edge of the tile falls along the edge of the previous tracing as shown. Trace around the tile again.
4. Repeat #3 until a full turn has been made.
  - a) What shape did you create?
  - b) How many times did you have to rotate the tile to create this shape?
5. Add colour and designs to the tessellation to make a piece of art.
6. How could you continue to use rotations to make a larger tessellation?

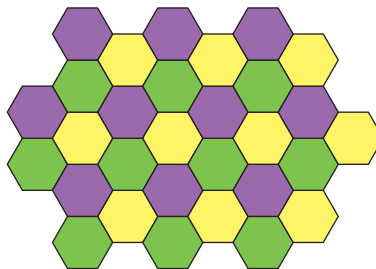


## Reflect on Your Findings

7. a) Describe how to use rotating polygons to create tessellations.
- b) What types of polygons can be used to make tessellations based on rotations? Explain.

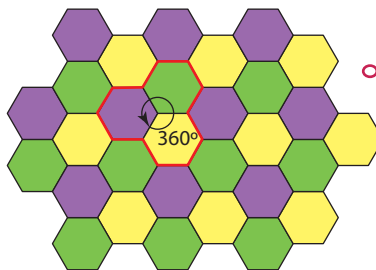
### Example: Identify the Transformation

What polygons and what transformation could be used to create this tessellation?



### Solution

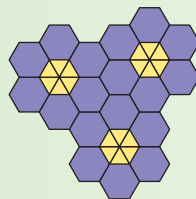
The tessellating tile is made up of a regular hexagon that has been rotated three times to make a complete turn. The three hexagons forming this tile can be translated horizontally and diagonally to enlarge the tessellation.



What other transformation(s) could create this tessellation?

### Show You Know

What polygons and transformations could be used to create this tessellation? Explain how you know.





## Key Ideas

- Tessellations can be made with two or more polygons as long as the interior angles where the polygons meet total exactly  $360^\circ$ .
- Rotations can be used to create tessellations.

### Communicate the Ideas

1. When creating a tessellation using rotations, why is it important for the sum of the angle measures at the point of rotation to equal  $360^\circ$ ? Explain.
2. Describe to a partner how to use rotating polygons to create tessellations.

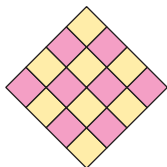
## Check Your Understanding

### Practise

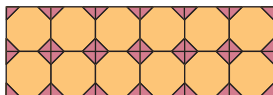
For help with #3 and #4, refer to the Example on page 458.

3. Identify the polygons used to create each tessellating tile.

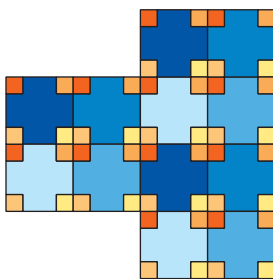
a)



b)



c)



4. What transformations could be used to create each tessellation in #3?

### Apply

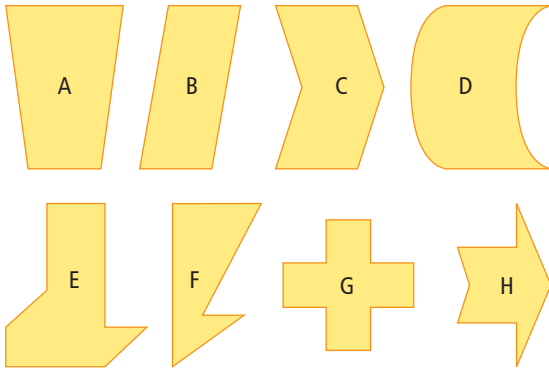
5. Examine the piece of stained glass.



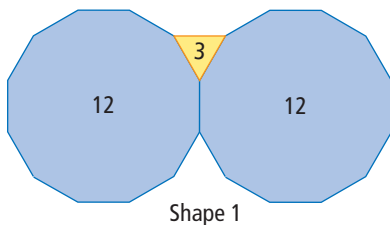
- a) Describe the transformation(s) used to make this pattern.
  - b) If you were using this pattern to tile the plane, what modifications would you have to make?
6. Design your own stained-glass window on grid paper. Describe the steps you followed to create the pattern.
  7. Create a tessellation using two different regular polygons and rotations.

## Extend

8. Which of the following shapes tessellate? Explain how you know a shape will or will not tessellate.

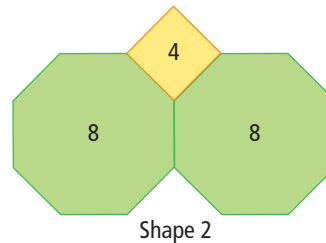


9. The diagram shows one arrangement of three or more polygons that can be used to create tessellations using rotations. One triangle and two dodecagons can be used because the angles at each vertex total  $360^\circ$  where they join. This is represented as (3, 12, 12). The table shows the features of this tessellation, for Shape 1.



| Tessellations Involving Three Regular Polygons | Shape 1                   | Shape 2 | Shape 3 | Shape 4 |
|--|---------------------------|---------|---------|---------|
| Triangle ( $60^\circ$ )                        | 1                         |         |         |         |
| Square ( $90^\circ$ )                          | 0                         |         |         |         |
| Pentagon ( $108^\circ$ )                       | 0                         |         |         |         |
| Hexagon ( $120^\circ$ )                        | 0                         |         |         |         |
| Octagon ( $135^\circ$ )                        | 0                         |         |         |         |
| Dodecagon ( $150^\circ$ )                      | 2                         |         |         |         |
| Number of Sides                                | (3, 12, 12)               |         |         |         |
| Sum of Angles                                  | $60 + 2(150) = 360^\circ$ |         |         |         |

- a) Copy the table into your notebook. Complete the table for Shape 2 for the diagram shown.



- b) Complete the table for Shapes 3 and 4, using different combinations of three or more regular polygons that total  $360^\circ$ .
- c) Create construction paper or cardboard cutouts of the regular polygons from part b). Try to tessellate the plane using the combinations that you believe will work.

## MATH LINK

Create your own pysanka design based on tessellating one or more polygons. Use at least one rotation in your design. Trace your design on grid paper, and colour it. Make sure it is the correct size to fit on an egg. If you have time, decorate an egg with your pysanka design.

### WWW Web Link

To see examples of pysankas, go to [www.mathlinks8.ca](http://www.mathlinks8.ca) and follow the links.

# 12.4

## Creating Escher-Style Tessellations

### Focus on...

After this lesson, you will be able to...

- ❑ create tessellations from combinations of regular and irregular polygons
- ❑ describe the tessellations in terms of the transformation used to create them



In the previous sections, you created tessellating patterns using regular and irregular polygons. When Escher created his tessellations, he did so in a variety of ways. Look at the two Escher works. What is different about the tessellations?

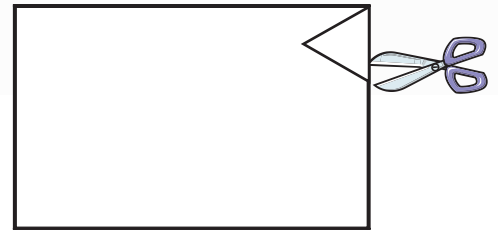
### Explore the Math

#### How do you make Escher-style tessellations?

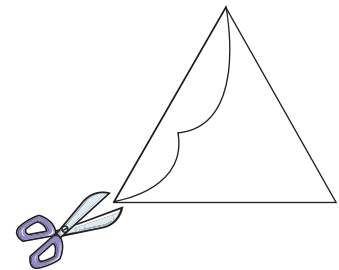
#### Materials

- ruler
- scissors
- glue stick
- cardboard or construction paper
- tape
- coloured pencils

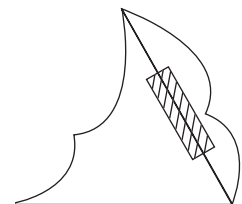
1. Draw an equilateral triangle with 6-cm sides on a blank piece of paper. Cut out the triangle and glue it to a sheet of cardboard or construction paper. Cut out the triangle again.



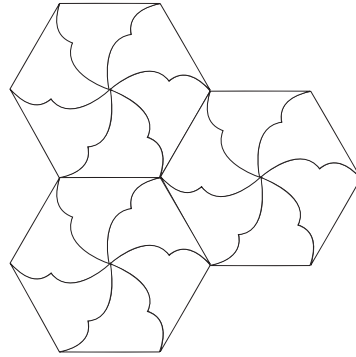
2. Inside the triangle, draw a curve that connects two adjacent vertices. Cut along the curve to remove a piece from one side of the triangle.



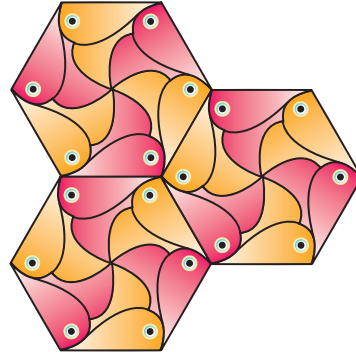
3. Rotate the piece you removed  $60^\circ$  counterclockwise about the vertex at the top end of the curve. This rotation moves the piece to another side of the triangle. Tape the piece in place to complete your tile.



4. To tessellate the plane, draw around the tile on a piece of paper. Then, rotate and draw around the tile over and over until you have a design you like.



5. Add colour and designs to the tessellation to make a piece of art.



6. Repeat steps 1 through 5 using a parallelogram and translations to create another Escher-style drawing.

### Reflect on Your Findings

7. You can use transformations to create Escher-style tessellations just as you did with regular and irregular polygons.
- Describe how to use rotations to create Escher-style tessellations.
  - What do you notice about the sum of the angle measures at the vertices where the tessellating tiles meet?
  - How does the area of the modified tile compare with the area of the original polygon? Explain.

#### Did You Know?

The leading geometer of the twentieth century was a professor at the University of Toronto named Donald Coxeter (1907–2003). He met M.C. Escher in 1954 and gave Escher some ideas for his art.

### Example: Identify the Transformation Used in a Tessellation

What transformation was used to create each of the following tessellations?



Tessellation A

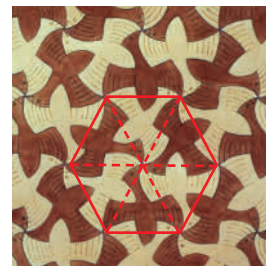


Tessellation B

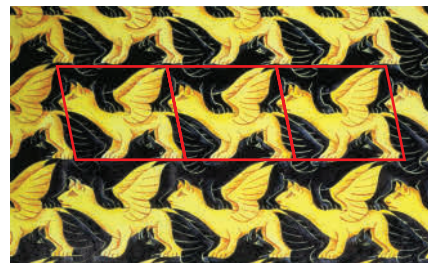


### Solution

Tessellation A is made up of triangles that have been rotated to form a hexagon. This tessellation is made using rotations.



Tessellation B is made up of figures that alternate gold to black and then repeat horizontally across the drawing. This tessellation is made using translations.



### Show You Know

What transformation was used to create this tessellation? Explain your answer.

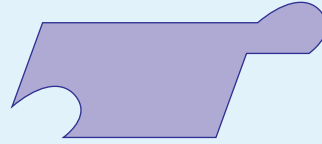


## Key Ideas

- You can create Escher-style tessellations using the same methods you used to create tessellations from regular or irregular polygons:
  - Start with a regular or irregular polygon.
  - The area of the tessellating tile must remain unchanged—any portion of the tile that is cut out must be reattached to the tile so that it fits with the next tile of the same shape.
  - Make sure there are no overlaps or gaps in the pattern.
  - Make sure interior angles at vertices total exactly  $360^\circ$ .
  - Use transformations to tessellate the plane.

## Communicate the Ideas

1. When creating a tile for an Escher-style tessellation, the original polygon is cut up. How do you know the area of the original polygon is maintained?
2. Rico believes that he can use this tile to create an Escher-style tessellation. Is he correct? Explain.
3. Tessellations must have no gaps or overlaps. What other two properties must be maintained when creating Escher-style tessellations?



## Check Your Understanding

### Practise

For help with #4 to #7, refer to the Example on pages 462–463.

4. Identify the transformations used to create each tessellation.

a)

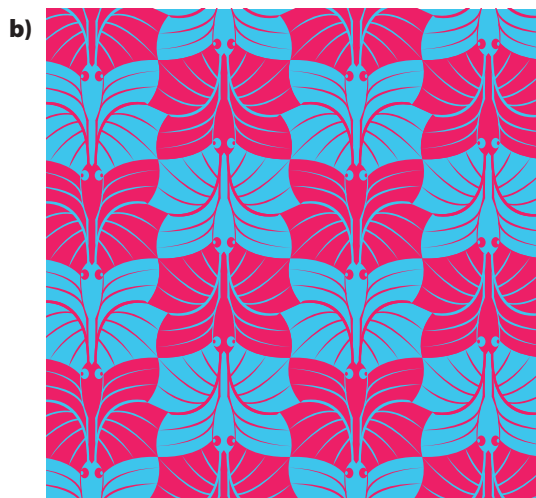
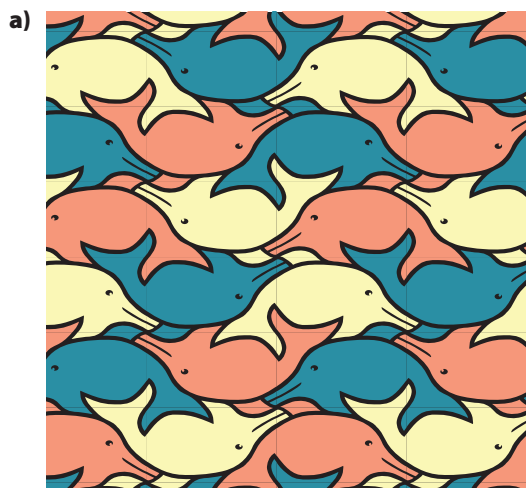
Digital rights not available.

b)

Digital rights not available.

5. Identify the original shape from which each tile was made for each tessellation in #4.

6. Identify the transformations used to create each tessellation.



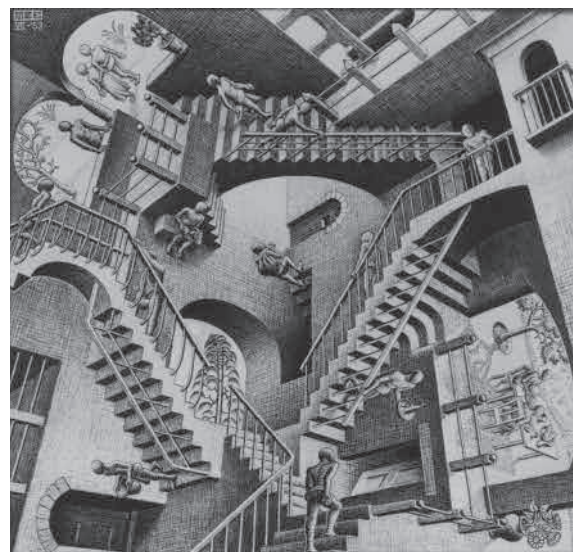
7. Identify the original shape from which each tile was made for each tessellation in #6.

### Apply

8. Create an Escher-style tessellation using a scalene triangle with translations.
9. Create an Escher-style tessellation using an equilateral triangle with rotations.
10. Create an Escher-style tessellation using squares with rotations and translations.

### Extend

11. Escher also used impossible figures in his art, as shown.



- a) What impossible figures were used in the drawing?
- b) Research other examples of Escher's art that include impossible figures.

## MATH LINK

Use an Escher-style tessellation to create a design for a binder cover, wrapping paper, a border for writing paper, or a placemat.

### WWW Web Link

To see examples of Escher's art, go to [www.mathlinks8.ca](http://www.mathlinks8.ca) and follow the links.

## Key Words

For #1 to #4, unscramble the letters for each puzzle. Use the clues to help you solve the puzzles.

## 1. LITGIN HTE EPALN

using repeated congruent shapes to cover a region without leaving gaps or overlapping

## 2. L E N A P

a 2-D flat surface that extends in all directions

## 3. S L T I O E E T A N L S

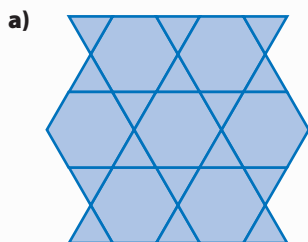
a pattern that covers a plane without overlapping or leaving gaps

## 4. R M T S A I N F N O T A O R

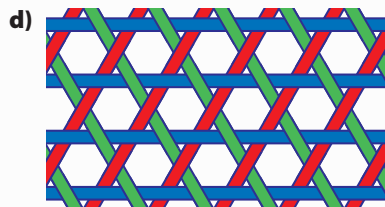
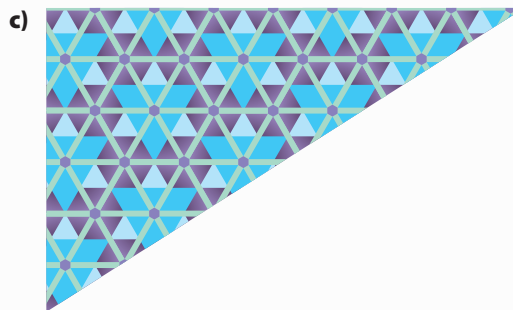
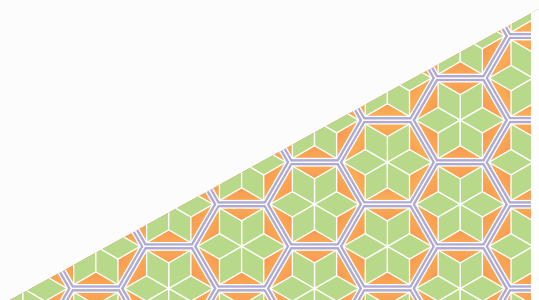
examples include translations, rotations, and reflections

### 12.1 Exploring Tessellations With Regular and Irregular Polygons, pages 446–451

5. What different types of polygons are used to create each tiling pattern?



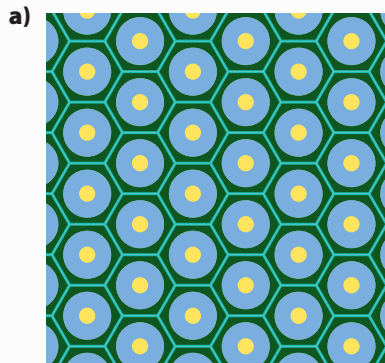
b)



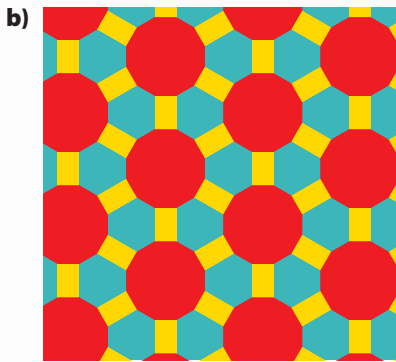
6. Which of the polygons in #5 are regular polygons and which are not? Explain how you know.
7. Can a regular octagon be used to tile the plane? Explain.

### 12.2 Constructing Tessellations Using Translations and Reflections, pages 452–456

8. What transformation or transformations could be used to create the following patterns?



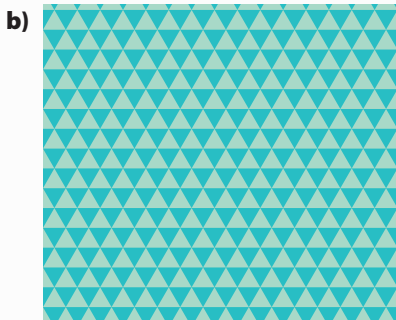
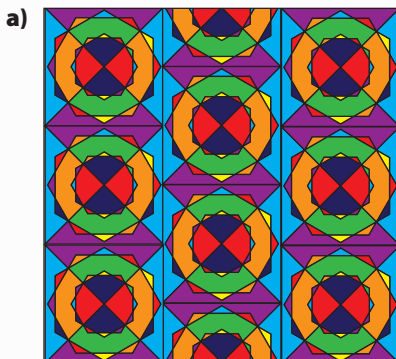




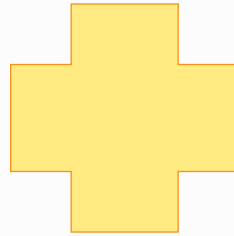
9. Create a tiling pattern using equilateral triangles and squares. Use one translation and one reflection to create the pattern.
10. Create a tessellation using three regular polygons and two different translations.

### 12.3 Constructing Tessellations Using Rotations, pages 457–460

11. What transformations could be used to create the following patterns?

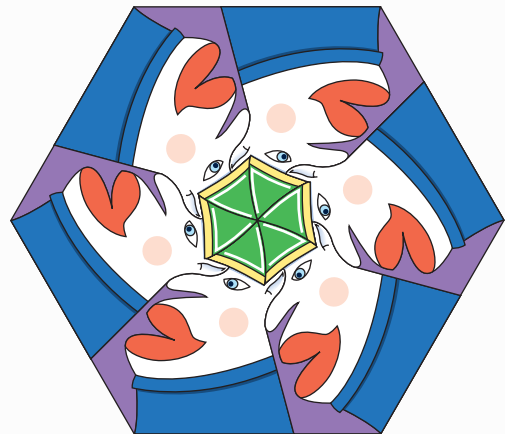


12. Create a tessellation using this irregular polygon. What additional polygon is needed to complete the pattern?



### 12.4 Creating Escher-Style Tessellations, pages 461–465

13. What original shape was used to create this tessellation?



14. What transformations were used to create the tessellation in #13?
15. Create two different Escher-style tessellations from the same square tile design. Use a different transformation for each tessellation.

For #1 to #5, select the best answer.

1. Which one of these regular polygons cannot be used to tile the plane?

A square  
 B triangle  
 C hexagon  
 D pentagon

2. When polygons can be used to create a tessellation, what is the sum of the interior angles where the vertices of the polygons meet?

A  $90^\circ$   
 B  $180^\circ$   
 C  $270^\circ$   
 D  $360^\circ$

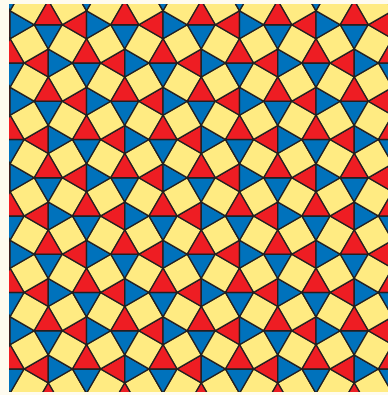
3. Which statement below is false?

A Any regular quadrilateral can be used to tessellate the plane.  
 B Any irregular quadrilateral can be used to tessellate plane.  
 C Any regular hexagon can be used to tessellate plane.  
 D Any irregular hexagon can be used to tessellate plane.

4. Which polygon can be used to create a tessellation?

A regular pentagon  
 B regular hexagon  
 C regular heptagon  
 D regular octagon

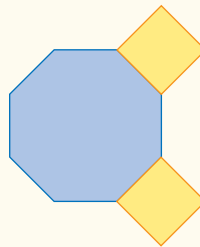
5. How many different polygons were used to create this tessellation?



A 1  
 B 2  
 C 3  
 D 4

### Short Answer

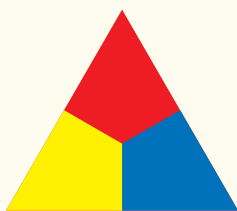
6. Can the regular octagon and two squares be used to tile the plane? Explain.



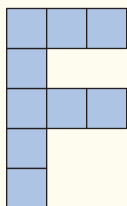
7. Can Jamie create a tessellation using this triangle? Explain.



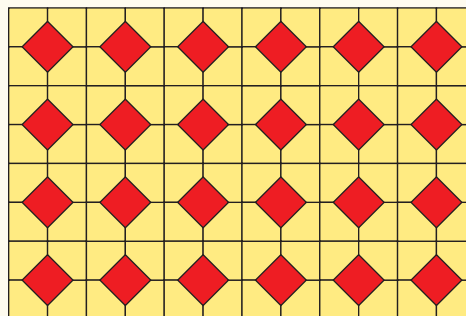
8. Describe how this pattern can be created. What type of polygon is being used to create the pattern?



9. Square tiles have been arranged in an F-shape. Use grid paper to find out if the F-shape will tile the plane.



10. Describe how you could create this tessellation.



### Extended Response

11. Create an Escher-style tessellation using an equilateral triangle or a square and rotations.

## WRAP IT UP!

Mosaic designs can be used on tiles, wallpaper, carpets, furniture, and fabrics.

- Create a mosaic design that incorporates at least two different shapes and two different transformations.
- Construct your mosaic using available materials, such as coloured construction paper, coloured transparencies, tile pieces, paints, etc.
- Write a brief paragraph describing the different shapes *and* transformations you used to create your mosaic.
- Work with other students to connect the patterns together to make a class mosaic.

# Math Games

## Playing at Tiling

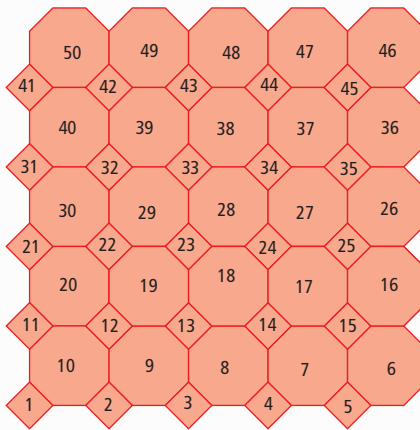
Many game boards, such as chess boards, are made from squares. Squares tessellate, so the board can be made without overlapping the squares or leaving gaps between them.

1. Game boards can be made from other polygons, or combinations of polygons, that tessellate. The board shown here includes squares and regular octagons.

Play a game on this board with a partner or in a small group.

These are the rules:


- Each player rolls one die to decide who will play first. If there is a tie, roll again.
  - For each turn, roll the two dice and identify the greater value. On the board, move your coloured counter that number of places ahead.
  - If you roll a double, move ahead to the next place that has a different shape from your present place. Then move ahead the number of places equal to the value from either die.
  - The first player to reach 50 wins.
2. Design a game board using a shape, or combination of shapes, that tessellates. Create the rules for a dice game to be played on this board. You might want to consider bonus points or penalty points for landing on a particular colour. Play the game with a partner or in a small group. Modify the rules to make the game better.



I rolled a 3 and a 5. The greater value is 5, so I moved my counter ahead 5 places on the board.

I rolled two 4s when my counter was on square 13. I moved ahead to the next octagon, number 16. I then moved ahead 4 places to position 20.

## Materials

- one Playing at Tiling game board per pair of students or small group 
- two standard 6-sided dice per pair of students or small group
- one coloured counter for each student



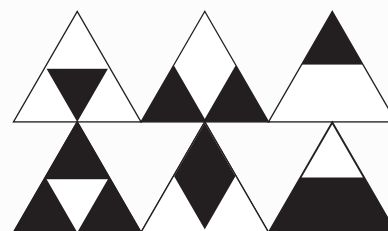
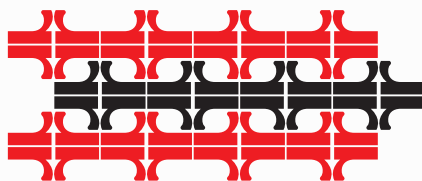
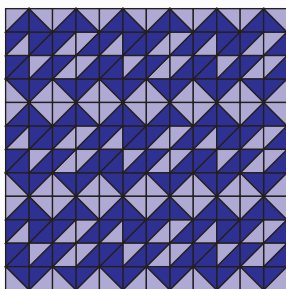
# Challenge in Real Life

## Border Design

Designers create patterns and border designs for such uses as tiles, wallpaper borders, upholstery, fabrics, and rugs. You have been commissioned to design and paint a border on the wall at the skateboard park. Using your knowledge of tessellations, create a design for a border 12 cm wide.

### Materials

- construction paper
- scissors
- coloured pencils or markers



1. On construction paper, design and cut out a regular polygon such as an equilateral triangle, a square, a pentagon, or a polygon with more than five sides. This is your template.
2. Use your template to create a reflection, rotation, and translation of your shape. Label each transformation.
3. Using your knowledge of transformations and your work from #2, create a border design on a piece of paper that is 12 cm  $\times$  28 cm. The design must use at least two different types of transformations.
4. Colour your design to emphasize the two types of transformations.

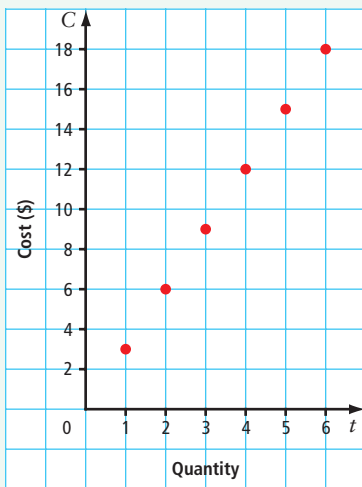
# Chapters 9–12 Review

## Chapter 9 Linear Relations

1. The table of values shows the number of triangles in an increasing pattern.

| Figure Number       | 1 | 2 | 3 | 4 |
|---------------------|---|---|---|---|
| Number of Triangles | 3 | 5 | 7 |   |

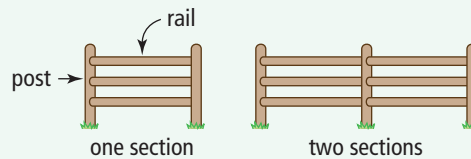
- a) How many triangles are in Figure 4?  
 b) Graph the table of values. Label your axes.  
 c) Does your graph represent a linear relation? Explain.
2. You buy a quantity of something according to the linear relation shown on the graph.



- a) Describe what you might have purchased. What is the cost if you purchase one item?  
 b) Describe patterns you see in the graph.  
 c) Make a table of values for the data on the graph. What variables might you use to label your table? Explain what the variables represent.

- d) What is an expression for the cost in terms of the quantity?  
 e) If the quantity is eight, what is the cost?

3. A farmer is building a post-and-rail fence around his yard. The number of rails in relation to the number of posts can be represented by  $r = 3p - 3$  where  $r$  is the number of rails and  $p$  is the number of posts.



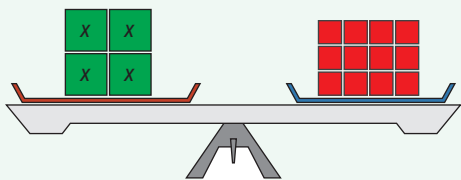
- a) Copy and complete the table of values.

| Number of Posts, $p$ | 2 | 3 | 4 | 5 | 6 | 7 |
|----------------------|---|---|---|---|---|---|
| Number of Rails, $r$ |   |   |   |   |   |   |

- b) Draw a labelled graph. Does the relationship appear to be linear? Explain.
4. Each of the following represents a linear relation.
- $$y = 2x - 3$$
- $$y = 2x + 1$$
- a) Make a table of values for each equation. Use at least five positive and five negative integer values for  $x$ . Why did you choose those  $x$ -values?  
 b) Graph both linear relations on the same grid.  
 c) What is similar about the graphs? What is different?

## Chapter 10 Solving Linear Equations

5. The diagram represents an equation.



- a) What equation does this diagram represent?  
b) What is the solution to the equation?

6. Use models or diagrams to solve each equation.

a)  $\frac{s}{2} = -5$

b)  $-3x + 6 = -3$

c)  $10 = 6 + \frac{v}{4}$

d)  $2(x - 5) = -4$

7. Solve each equation. Check your answers.

a)  $\frac{x}{7} = -4$

b)  $14 = -26 + 5x$

c)  $11 - \frac{x}{3} = 17$

d)  $4(x - 9) = -16$

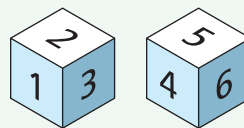
8. Jason's age is three years fewer than  $\frac{1}{3}$  his father's age. Jason is ten years old.

- a) What equation models this situation?  
b) How old is Jason's father?

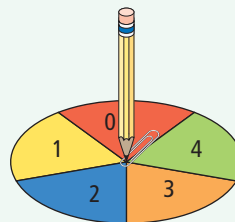
9. Elijah works in a diamond mine. When he works the late shift, \$2/h is added to his regular hourly wage. Last week, he worked the late shift for a total of 40 h and made \$960. Write and solve an equation to determine Elijah's regular hourly wage.

## Chapter 11 Probability

10. Two six-sided dice are rolled.

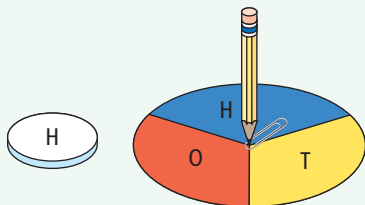


- a) Draw a tree diagram or table to represent the sample space.  
b) What is the probability that an even number is rolled on both dice? Express the probability as a fraction.  
c) What is the probability that the sum of the two numbers is greater than or equal to six,  $P(\text{sum} \geq 6)$ ? Express the probability as a fraction.
11. A spinner is divided into five equal regions and labelled with the whole numbers zero to four.



- a) What is the probability of spinning an odd number?  
b) What is the probability of spinning an even number?  
c) If you spin the spinner twice, what is the probability of spinning an odd number on the first spin and an even number on the second spin?
12. An online computer company has a sale in which customers choose one of four different computers and one of three different printers. How many computer-printer options are available?

13. Gillian flips a disk labelled H on one side and T on the other side. She spins a spinner divided into three equal regions once.

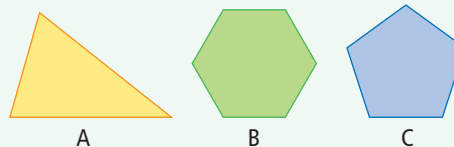


- a) What is the probability that an H is flipped on the disk? What is the probability that an H is spun on the spinner? Express the probabilities as fractions.
- b) What is the probability of H appearing on both the disk and spinner? Express the probability as a fraction.
- c) Check your answer to part b) using another method.
14. Ria and Renata are identical twins. They like the same kind of cereal and their favourite colour is blue. The cereal company that makes their favourite cereal is having a promotion—each box of cereal includes a spinning top toy that comes in one of four colours, including blue. The girls want to run a simulation.
- a) Describe a simulation that the twins could run to determine the probability that the next two boxes of cereal they open will each contain a blue spinning top.
- b) What assumption(s) do Ria and Renata need to make regarding the spinning tops?
- c) In a set of 20 trials, two blue tops resulted only once. What is the experimental probability for this experiment? Express your answer as a fraction and as a percent.

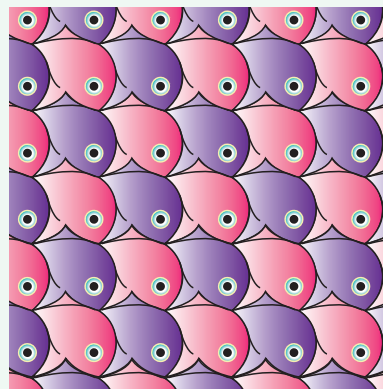
- d) What is the theoretical probability that the next two boxes of cereal will contain a blue spinning top? Express your answer as a fraction and as a percent.

## Chapter 12 Tessellations

15. Which of the following polygons cannot be used to tile the plane? Explain how you know.



16. Create a tiling pattern using a square and one other shape. Describe the transformation(s) you used to create your pattern.
17. Describe how this tessellation can be created.



18. Create an Escher-style tessellation using a parallelogram as the original shape. Describe the transformation(s) you used for your tessellation.




# Task

## Put Out a Forest Fire

One effective way to fight a forest fire is to drop water and fire retardant on it from an airplane. A number of factors influence how effective this is, including wind direction and speed, speed of the airplane, and temperature of the fire. You are training as a pilot of a firefighting airplane. Create a simulation to observe how effective you can be at putting out the fire.

1. Do the following to prepare your simulation.
  - Draw a rectangle that is 14 cm by 16 cm on a blank sheet of paper.
  - Cut out the shape you have been given. The full triangle counts as two shapes. In order to fill your rectangle, you will sometimes use half of the triangle. Each half triangle counts as one shape.
  - Using transformations and the shape provided, tile your paper until the rectangle is completely full.
  - Colour the shapes in your tessellation using the ratio of 1 blue : 3 orange : 4 green.
  - Cut out the rectangle.
  - Work with a group of at least three other students. Join your tessellated paper with those of the other group members. This larger tessellation represents the map of a forest fire. Orange represents the area that is burning, green is the forest, and blue is the lakes.
2. The object of the simulation is to put out the entire fire by dropping water on each of the orange areas.
  - One at a time, stand beside the tessellated map and drop three pieces of modelling clay or three chips onto it. Each drop represents a water drop.
  - Record what colour each drop lands on.
3. a) Hitting an orange shape puts out the fire in that part of the tessellation, including all of the orange shapes that are attached to the orange shape that was hit. What is the experimental probability of hitting an orange shape?  
b) How much did the simulation help you improve your understanding of experimental probability versus theoretical probability? Explain why.  
c) What is the theoretical probability of randomly hitting an orange shape?

### Materials

- shape to tessellate 
- ruler
- coloured pencils (orange, green, blue)
- modelling clay or bingo chips

