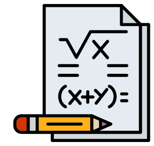


Area & Volume Models

Area Models (2D)

In our activity, what did you notice when creating perfect squares and rectangles?

⇒ The sides of a rectangle have ***different lengths*** and opposite sides are equal in length.

⇒ The sides of the perfect squares were ***all equal***.

Every square is a rectangle? True / False

Every rectangle is a square? True / False

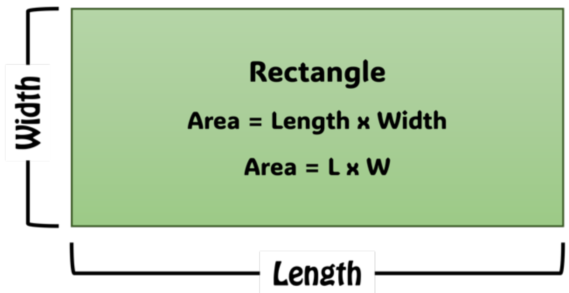
Investigate

Using the grid below make as many different rectangles as you can with the following areas: 4 square units | 6 square units | 8 square units | 9 square units | 12 square units. Draw each of your rectangles on the grid paper below.

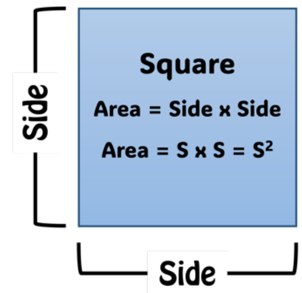
Circle which of the above areas you were able to make a ***square***.

How is the side length of a square and rectangle related to its area?

The area is equal to its side length multiplied by its other side width.



But for a square, each side is the same length, therefore we get...



When we multiply a number by the same number, we call it *Squaring* the number.

Example:

⇒ The square of 4 is 16, we would say "*4 squared is 16*"

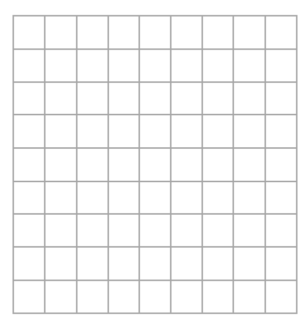
⇒ $5^2 = 5 \times 5 = 25$, we would say "*5 squared is 25*"

We can model a square number by drawing a square whose area is equal to the square of a given number.

Square numbers can be shown by using diagrams, symbols, or words!

Example: Show that 49 is a square number. Use a diagram, symbols and words.

Draw a square with area of 49 square units. Start by drawing a single dot in a grid and start placing additional dots in a reverse "L" direction.



The side length of the square would be 7 units

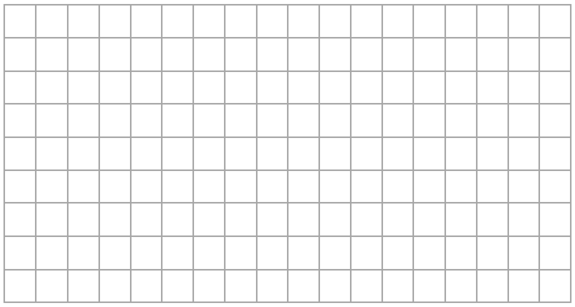
Using symbols we would write $7 \times 7 = 7^2 = 49$

Using words, we would say "*7 squared is 49*"

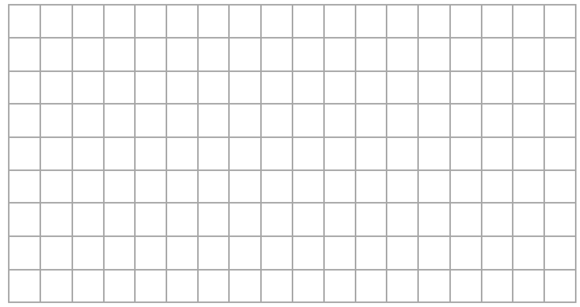
Important Note: Whenever we talk about the area, the units of measure are expressed as ***units squared*** (e.g. cm^2, m^2, km^2, ft^2, etc....)

Practice:
What is the area of the following shapes? Draw and label the shape on the grid provided. Which one has the largest area?

A rectangle with a length of 9 cm and a width of 7 cm?



A square with a side length of 8 cm?



Volume Models (3D)

In our activity, you should have noticed distinct characteristics when constructing perfect cubes and rectangular prisms:

Perfect Cubes:

⇒ Perfect cubes are three-dimensional shapes where ***all edges*** (length, width, and height) ***are of equal length***.

⇒ Each ***face*** of a perfect cube is a ***perfect square***.

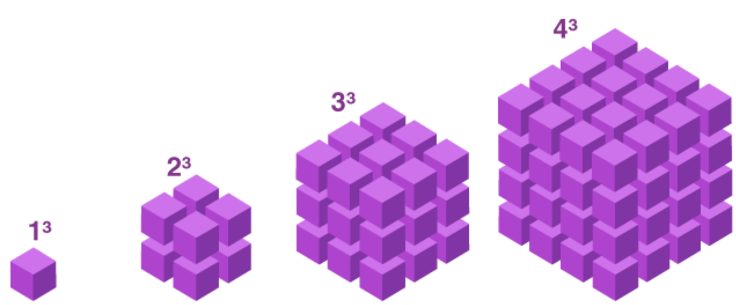
Rectangular Prisms:

⇒ Rectangular prisms are also three-dimensional shapes characterized by six faces (3-pairs), each of which is a rectangle.

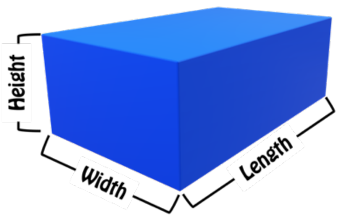
⇒ Unlike cubes, rectangular prisms have ***different lengths***, widths, and heights, allowing for varying dimensions along each axis.

Every cube is a rectangular prism? True / False
Every rectangular prism is a cube? True / False

How is the side length of a cube and rectangular prism related to its volume?

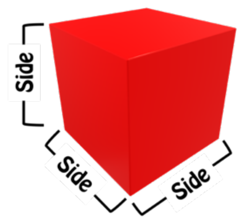


The volume is equal to its side length multiplied by its side width and side height.



Rectangular Prism
Volume = Length x Width x Height
Volume = L x W x H

But for a Cube, each side is the same length, therefore we get…



Cube
Volume = Side x Side x Side
Volume = $S \times S \times S = S^3$

When we multiply a number by the same number twice, we call it ***Cubing*** the number.

Example:

⇒ The cube of 4 is **64**, we would say "***4 cubed is 64***"

⇒ $5^3 = 5 \times 5 \times 5 =$ **125**, we would say "***5 cubed is 125***"

Important Note: Whenever we talk about the volume, the units of measure are expressed as ***units cubed*** (e.g. cm^3, m^3, km^3, ft^3, etc.…)

Practice:

What is the volume of the following shapes? Draw and label the shape on the grid provided. Which one has the largest volume?

A rectangular prism with a length of 9 m, a width of 4 m and height of 7 m?

A cube with a side length of 6 m?

Perfect Squares and Square Roots

Recall a ***factor*** is a number that divides another number without leaving a remainder (i.e. no decimal). In other words, if you multiply a factor by another number, you get the original number as the result.

Investigate

The following chart shows the factors of each whole number from 1 to 30.

																							24						30
																							12						15
											12						18		20				8				28		10
											6				16		9		10				6				14		6
					6		8		10		4		14	15	8		6		5	21	22		4		26	27	7		5
			4		3		4	9	5		3		7	5	4		3		4	7	11		3	25	13	9	4		3
	2	3	2	5	2	7	2	3	2	11	2	13	2	3	2	17	2	19	2	3	2	23	2	5	2	3	2	29	2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**

Which numbers have <u>only two</u> factors? What do you notice about these numbers?

⇒ Numbers that have only two factors are ***Prime Numbers***. They are divisible by only 1 and themselves.

Which numbers have an <u>odd</u> number of factors?

⇒ Since factors come in <u>pairs</u>, these are numbers that must have two of the same factors and, therefore must be ***perfect squares*** or **square numbers**!

⇒ The middle factor is the ***Root number***!

Determining Perfect Squares

You can tell if a number is a perfect square a couple of different ways.

Method 1: Using Division:

If we find that in a division sentence, the quotient is equal to the divisor, the number is a perfect square.

Example: $36 \div 6 = 6$

Method 2: Using Factors:
If we find that a number has an *ODD* number of factors, then it is a square number.

Example: 25 has 3 factors: 1, 5, and 25.

Method 3: Using Prime Factorization:
If we find that a number has an *EVEN* number of prime factors, then it is a square number.

Example: Prime factors of 25 are 5 and 5…so 5^2
Prime factors of 36 are 2, 2, 3 and 3…so 2^2 and 3^2

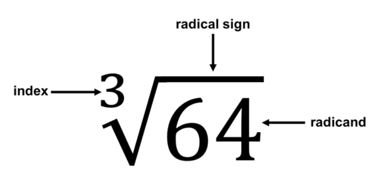
Method 4: Using Memory
You can memorize the list of common perfect squares or use your multiplication facts.

Perfect Squares to Memorize:	$0^2 = 0$	$4^2 = 16$	$8^2 = 64$	$11^2 = 121$
	$1^2 = 1$	$5^2 = 25$	$9^2 = 81$	$12^2 = 144$
	$2^2 = 4$	$6^2 = 36$	$10^2 = 100$	$13^2 = 169$
	$3^2 = 9$	$7^2 = 49$		

Square Roots
A square root (√) is a number which, when squared, results in a given number.

⇒ 3 is a square root of 9, so we write it as: $3 = \sqrt{9}$
⇒ The square root of 16 is 4, so we write is as: $\sqrt{16} = 4$.

FYI: square roots, and any root in general is called a ***radical!*** More on this in Grade 9 and up, but for now just understand the parts.



Taking the square root of a number is the opposite operation of squaring a number, just like subtraction is the opposite of addition and division is of the opposite multiplication.

Squaring and taking the square root are ***inverse*** operations. They undo each other!

⇒ $3^2 = 9 \rightarrow \sqrt{9} = 3$
⇒ $4^2 = 16 \rightarrow \sqrt{16} = 4$

Taking a square root will work on any number, not just perfect squares.

What about Negative Integers?

Well, let's think about this…

$6^2 = 36$ & $(-6)^2 = 36$….What happens when try and go backwards $\sqrt{36}$ = ???

How do we know if the root was +6 or -6? Well, we don't, so to overcome this we use the following notion **+/-** or **±** to denote the answer could be either (+) or (-)

Therefore, the correct answer for $\sqrt{36} = \pm 6$

In some cases, the negative (-) answer maybe illogically, such as in a word problem; negative lengths, negative items, negative time, negative items, etc…in these cases it is appropriate to express only the positive (+) answer.

Caveat: This doesn't not apply in the reverse!

YOU CAN NOT TAKE A SQUARE ROOT OF A NEGATIVE NUMBER

Let us consider the following: $\sqrt{(-16)}$

This means that the product of ***two*** identical numbers produces a negative (-) value. But we know from our integer rules theh only way to make a negative (-) value is by multiplying a positive (+) number with a negative (-) number…therefore the numbers are not identical.

$(+4) \times (+4) = +16 \rightarrow (+4)^2 = +16$
$(-4) \times (-4) = +16 \rightarrow (-4)^2 = +16$
$(+4) \times (-4) = -16 \rightarrow$ Bases are not the same, therefore no repeated multiplication, hence cannot be expressed as an exponent

However, you can have a negative outside (i.e. in front) of the square root. You can think of this as the square root being multiplied by negative one (-1)

Example:

a) $-\sqrt{9}$ =________ b) $-\sqrt{36}$ = ________ c) $-\sqrt{(-81)}$ = ________

Non-Perfect Squares and Square Roots

What if the number is **NOT** a Perfect Square number?

Estimating Square Roots

When dealing with non-perfect square numbers like 17 or 27, finding their square roots can be a bit challenging.

Method 1: Visually

A helpful way to estimate square roots is by visually constructing squares like we did previously by drawing dots.

Step 1: Start by drawing 1 dot.
Step 2: Continue drawing dots in a backward "L" direction.
Step 3: Continue adding dots until you run out.
Step 4: Count the number of dots on each side of your array. The smaller value will tell you what whole number your value will start from.
Step 5: In your incomplete "L" count the number of dots you have.
Step 6: Take the number dots you have divided by the total number for dots required to complete the "L".
Step 7: Add this value as the decimal after your value determined in step 4.

Practice: Estimate the square roots for the following using the visual method.

d) $\sqrt{19}$ =________ e) $\sqrt{31}$ = ________ f) $\sqrt{47}$ = ________

Method 2: Benchmarking a Root (Number Line)

In this approach, you identify the known lower and upper perfect squares and the difference between them and the root you would like to determine.

Step 1: Identify Lower and Upper perfect square. The lower value will tell you what whole number your value will start from.
Step 2: Determine the difference between the upper and lower values. This will be your denominator value.
Step 3: Determine the difference between the root you are trying to determine with either the lower OR upper limit. This will be your numerator.
Step 4: Divide your result from step 3 by your result in step 2 and add this value to your starting value in Step 1.

Practice: Estimate the square roots for the following using the benchmark method.

a) $\sqrt{19}$ =________ b) $\sqrt{31}$ = ________ c) $\sqrt{47}$ = ________

Using your Calculator!

Your calculator is equipped with a magically button square root button, that can do all of this for you!

Practice:
Use your calculator to determine the answer to the following to a maximum of 4 decimal places if required.

a) $\sqrt{81}$ =________ b) $\sqrt{(-16)}$ = ________ c) $-\sqrt{9}$ = ________

d) $\sqrt{1}$ =________ e) $\sqrt{(-36)}$ = ________ f) $-\sqrt{29}$ = ________

g) $-\sqrt{45}$ =________ h) $\sqrt{2}$ = ________ i) $-\sqrt{136}$ = ________

j) $\sqrt{576}$ =________ k) $\sqrt{2116}$ = ________ l) $\sqrt{46225}$ = ________

Cubed Roots

The process is much the same for cubed root but granted more difficult. So, for cubed roots we only use our calculator!

Your calculator will be equipped with a button that looks like: _______________

Unlike square roots, a cubed root could negative (-) inside of the root sign.

Let us consider the following: $\sqrt{(-27)}$

This means that the product of ***three*** identical numbers produces a negative (-) value. We know from our integer rules the only way to make a negative (-) is by multiplying an odd number of negative (-) values…

$(+3) \times (+3) \times (+3) = +29 \rightarrow (+3)^3 = +27$
$(-3) \times (-3) \times (-3) = -27 \rightarrow (-3)^3 = -27$ Bingo!
$(-3) \times (+3)$ X $(+3) = -27 \rightarrow$ Bases are not the same, therefore no repeated multiplication, hence cannot be expressed as an exponent

However, you can have a ***negative outside*** (i.e. in front) of the square root. You can think of this as the square root being ***multiplied by negative one*** (-1)

Example:

a) $\sqrt[3]{(-8)}$ =________ b) $-\sqrt[3]{(-64)}$ = ________ c) $\sqrt[3]{(-125)}$ = ________

Practice:

Use your calculator to determine the answer to the following to a maximum of 4 decimal places if required.

a) $\sqrt[3]{120}$ =________ b) $\sqrt[3]{(-9)}$ = ________ c) $-\sqrt[3]{27}$ = ________

d) $\sqrt[3]{1}$ =________ e) $\sqrt[3]{(-216)}$ = ________ f) $-\sqrt[3]{(-64)}$ = ________

g) $\sqrt[3]{45}$ =________ h) $\sqrt[3]{20}$ = ________ i) $-\sqrt[3]{343}$ = ________

BONUS: Algorithm Trick! (Only works for Perfect Squares)
Here is a video of the trick: [How to Calculate Square Roots]

Step 1: Start by memorizing the squares of numbers from 0 to 9.
Step 2: Look at the last digit of the number whose square root you want to find.
Step 3: Match the last digit to the squares you've memorized. If the last digit matches a square's last digit, that square is your starting point.
Step 4: Cross out the last two digits of the number you're finding the square root for.
Step 5: Look for the square closest to the remaining number without exceeding it.
Step 6: If there are multiple possibilities, use the remaining digits to determine the correct square. You may have to test the squares in between to find the correct one.
Step 7: Once you've found the correct square, write down the corresponding digit.

a) $\sqrt{729}$ =_______ b) $\sqrt{4096}$ = _______ c) $\sqrt{7921}$ = _______

This trick also works similarly for perfect cubes.

Simplifying Roots

In math we typically don't like decimal approximations and prefer an exact value. To do this we find the prime factors of value inside the root and use the properties of square roots to simplify our answer.

Let's consider the following example: $\sqrt{40}$

What are the prime factors of 40? ________________________

Replace 40 with the product of all the prime factors: $\sqrt{(2 \times 2 \times 2 \times 5)}$

Use the "Jail Break" Method: for every *PAIR* of identical factors one factor escapes in front of the square root sign and the other dies escaping ☹

$$\text{So } \sqrt{40} = \sqrt{(2 \times 2 \times 2 \times 5)} = 2\sqrt{(2 \times 5)} = 2\sqrt{10}$$

Examples:

a) $\sqrt{24}$ =________ b) $\sqrt{54}$ = ________ c) $\sqrt{250}$ = ________

Cubed Roots

Similarly, we can apply the same method for cubed roots; however, this time we need *THREE* identical factors in order to pull on out.

Examples:

a) $\sqrt[3]{24}$ =________ b) $\sqrt[3]{54}$ = ________ c) $\sqrt[3]{250}$ = ________

Practice:

Simplify the following.

a) $\sqrt{98}$ =________ b) $\sqrt{30}$ = ________ c) $\sqrt[3]{88}$ = ________

d) $\sqrt{63}$ =________ e) $\sqrt[3]{16}$ = ________ f) $\sqrt[3]{40}$ = ________

g) $\sqrt[3]{48}$ =________ h) $\sqrt{20}$ = ________ i) $\sqrt[3]{32}$ = ________

j) $\sqrt{48}$ =________ k) $\sqrt[3]{128}$ = ________ l) $\sqrt[3]{144}$ = ________